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The Deformed Trigonometric Functions of two Variables

Sladjana Marinković [1], *Miomir Stanković* [2], *Edin Mulalić* [3]

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Recently, various generalizations and deformations of the elementary functions were introduced. Since a lot of natural phenomena have both discrete and continual aspects, deformations which are able to express both of them are of particular interest. In this paper, we consider the trigonometry induced by one parameter deformation of the exponential function of two variables $e_h(x,y) = (1+hx)^{y/h}$ $(h \in \mathbb{R}\setminus\{0\},\ x \in \mathbb{C}\setminus\{-1/h\},\ y \in \mathbb{R})$. In this manner, we define deformed sine and cosine functions and analyze their various properties. We give series expansions of these functions, formulas which have their similar counterparts in the classical trigonometry, and interesting difference and differential properties.

MSC 2010: 33B10, 33E20

Key Words: exponential function, trigonometric functions, deformed exponential function

1. Preliminaries

Among a lot of deformations of the exponential function proposed recently by many authors, we can emphasize Tsallis's q–exponential function ([6], [7])

$$e_q^x = (1+(1-q)x)^{1/(1-q)} \quad (q \neq 1,\ x \in \mathbb{R},\ 1+(1-q)x > 0),$$

which is the ground of his formalism of statistical mechanics. In [4], a new deformation of the exponential function of two variables was introduced with the purpose to join both suitable difference and differential properties. So, for $h \in \mathbb{R}\setminus\{0\}$, the function $(x,y) \mapsto e_h(x,y)$ is defined by

$$e_h(x,y) = (1+hx)^{y/h} \qquad (x \in \mathbb{C}\setminus\{-1/h\},\ y \in \mathbb{R}). \tag{1}$$

Since $\lim_{h\to 0} e_h(x,y) = e^{xy}$, this function can be viewed as a one–parameter deformation of the exponential function of two variables. If $h = 1-q$ $(q \neq 1)$ and $y = 1$, the function (1) becomes the Tsallis q–exponential function, i.e., $e_{1-q}(x,1) = e_q^x$.

Notice that function (1) can be written in the form

$$e_h(x,y) = e^{\{x\}_h\, y}, \tag{2}$$

where the deformation

$$\{x\}_h = \frac{1}{h}\ \ln(1+hx) \qquad (x \in \mathbb{C} \setminus \{-1/h\}) \tag{3}$$

is used.

Proposition 1.1 [4] *For the function $(x,y) \mapsto e_h(x,y)$ the following holds:*

$$\begin{aligned}
e_h(x,y) &> 0 \qquad (x < -1/h \text{ for } h<0 \quad \text{or} \quad x > -1/h \text{ for } h>0),\\
e_h(0,y) &= e_h(x,0) = 1,\\
e_{-h}(x,y) &= e_h(-x,-y) \qquad (x \neq 1/h), && (4)\\
e_h(x, y_1+y_2) &= e_h(x,y_1)e_h(x,y_2), && (5)\\
e_h(x_1 \oplus_h x_2, y) &= e_h(x_1,y)e_h(x_2,y), && (6)\\
e_h(x_1 \ominus_h x_2, y) &= e_h(x_1,y)e_h(x_2,-y), && (7)
\end{aligned}$$

where

$$x_1 \oplus_h x_2 = x_1 + x_2 + hx_1x_2, \qquad x_1 \ominus_h x_2 = \frac{x_1 - x_2}{1+hx_2} \quad (x_2 \neq -1/h). \tag{8}$$

Notice that

$$x \oplus_h (-x) = -hx^2, \qquad e_h(x \oplus_h (-x), y) = e_h(-hx^2, y).$$

For the function $(x,y) \mapsto e_h(x,y)$ the following representations hold:

$$e_h(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n y^{(n,h)} = \sum_{n=0}^{\infty} \frac{1}{n!} \{x\}_h^n\ y^n \qquad (|hx| < 1), \tag{9}$$

where the backward generalized integer powers of a real number are given by

$$z^{(0,h)} = 1, \qquad z^{(n,h)} = z(z-h)(z-2h)\cdots(z-(n-1)h).$$

Let us recall the h–difference operator of the first order:

$$\Delta_{z,h} f(z) = \frac{f(z+h) - f(z)}{h} = \frac{1}{h}(E_h - I) f(z), \tag{10}$$

where I is identity and E_h is shift–operator, and more general, the h–difference operator of the order $\alpha > 0$:

$$\Delta^{\alpha}_{z,h} f(z) = \frac{1}{h^{\alpha}} \sum_{k=0}^{+\infty} (-1)^k \binom{\alpha}{k} f(z + (\alpha - k)h).$$

The following equalities are valid:

$$\Delta^{\alpha}_{y,h} e_h(x,y) = x^{\alpha} e_h(x,y), \qquad \frac{\partial}{\partial y} e_h(x,y) = \{x\}_h \; e_h(x,y). \tag{11}$$

2. Deformed trigonometric functions: Definitions and basic properties

In [1] a deformation of the trigonometric and hyperbolic functions in the framework of the Tsallis exponentials is presented. Attending the deformation of q–exponential function (1), we can define other deformed trigonometric functions of two variables.

Let $h \in \mathbb{R} \setminus \{0\}$. For $x \in \mathbb{C} \setminus \{-i/h, i/h\}$ and $y \in \mathbb{R}$, we define the functions $(x,y) \mapsto \cos_h(x,y)$ and $(x,y) \mapsto \sin_h(x,y)$ by

$$\cos_h(x,y) = \frac{e_h(ix,y) + e_h(-ix,y)}{2}\ , \tag{12}$$

$$\sin_h(x,y) = \frac{e_h(ix,y) - e_h(-ix,y)}{2i}\ . \tag{13}$$

The following limit behavior is evident:

$$\lim_{h\to 0} \cos_h(x,y) = \cos(xy), \qquad \lim_{h\to 0} \sin_h(x,y) = \sin(xy).$$

Here we give two analogues of the basic trigonometry identity.

Proposition 2.1. *For $x \in \mathbb{C} \setminus \{-i/h, i/h\}$ and $y \in \mathbb{R}$ the following equalities hold:*

$$\sin^2_h(x,y) + \cos^2_h(x,y) = e_h(hx^2, y),$$

$$\sin_h(x,y) \sin_{-h}(x,y) + \cos_h(x,y) \cos_{-h}(x,y) = 1.$$

Proof. The first identity follows immediately from the definition:

$$\begin{aligned}\sin_h^2(x,y)+\cos_h^2(x,y) &= e_h(ix,y)e_h(-ix,y)=e_h(ix\oplus_h(-ix),y)\\ &= e_h(hx^2,y).\end{aligned}$$

According to (4), we get the second one by varying the sign of the parameter h. ■

The deformed trigonometric functions satisfy some similar equalities as the usual trigonometric functions.

Theorem 2.1. *For $x\in\mathbb{C}\setminus\{-i/h,i/h\}$ and $y_1,y_2\in\mathbb{R}$ the following equalities hold:*

- $\cos_h(x,y_1\pm y_2)=\cos_h(x,y_1)\cos_h(x,y_2)\mp\sin_h(x,y_1)\sin_h(x,y_2)$
- $\sin_h(x,y_1\pm y_2)=\sin_h(x,y_1)\cos_h(x,y_2)\pm\cos_h(x,y_1)\sin_h(x,y_2)$
- $2\sin_h(x,y_1)\cos_h(x,y_2)=\sin_h(x,y_1+y_2)+\sin_h(x,y_1-y_2)$
- $2\cos_h(x,y_1)\cos_h(x,y_2)=\cos_h(x,y_1+y_2)+\cos_h(x,y_1-y_2)$
- $2\sin_h(x,y_1)\sin_h(x,y_2)=\cos_h(x,y_1+y_2)-\cos_h(x,y_1-y_2)$
- $\sin_h(x,y_1)\pm\sin_h(x,y_2)=2\sin_h(x,\frac{y_1\pm y_2}{2})\cos_h(x,\frac{y_1\mp y_2}{2})$
- $\cos_h(x,y_1)+\cos_h(x,y_2)=2\cos_h(x,\frac{y_1+y_2}{2})\cos_h(x,\frac{y_1-y_2}{2})$
- $\cos_h(x,y_1)-\cos_h(x,y_2)=2\sin_h(x,\frac{y_1+y_2}{2})\sin_h(x,\frac{y_1-y_2}{2})$.

Proof. Here, we are giving proof of the first identity only. Other identities can be proved in similar manner. From the definition of the deformed trigonometric functions we have

$$e_h(ix,y)=\cos_h(x,y)+i\sin_h(x,y),\quad e_h(-ix,y)=\cos_h(x,y)-i\sin_h(x,y).$$

Using the previous two identities and equality (5), we obtain

$$\begin{aligned}\cos_h(x,y_1+y_2) &= \frac{1}{2}(e_h(ix,y_1)e_h(ix,y_2)+e_h(-ix,y_1)e_h(-ix,y_2))\\ &= \frac{1}{2}\Big((\cos_h(x,y_1)+i\sin_h(x,y_1))(\cos_h(x,y_2)+i\sin_h(x,y_2))\\ &\quad+(\cos_h(x,y_1)-i\sin_h(x,y_1))(\cos_h(x,y_2)-i\sin_h(x,y_2))\Big)\\ &= \cos_h(x,y_1)\cos_h(x,y_2)-\sin_h(x,y_1)\sin_h(x,y_2).\end{aligned}$$ ■

3. Difference and differential properties

In this section we consider the results of some operators on the deformed trigonometric functions. First, we look for the difference operators.

Theorem 3.1. *For $\alpha > 0$ the following holds:*

$$\begin{aligned}\Delta^{\alpha}_{y,h}\cos_h(x,y) &= x^{\alpha}\Big(\cos\frac{\alpha\pi}{2}\cos_h(x,y) - \sin\frac{\alpha\pi}{2}\sin_h(x,y)\Big),\\ \Delta^{\alpha}_{y,h}\sin_h(x,y) &= x^{\alpha}\Big(\cos\frac{\alpha\pi}{2}\sin_h(x,y) + \sin\frac{\alpha\pi}{2}\cos_h(x,y)\Big).\end{aligned}$$

P r o o f. In view of the linearity of the operator $\Delta^{\alpha}_{y,h}$ and equality (11), we have

$$\begin{aligned}\Delta^{\alpha}_{y,h}\cos_h(x,y) &= \frac{1}{2}(\Delta^{\alpha}_{y,h}e_h(ix,y) + \Delta^{\alpha}_{y,h}e_h(-ix,y))\\ &= \frac{1}{2}((ix)^{\alpha}e_h(ix,y) + (-ix)^{\alpha}e_h(-ix,y))\\ &= \frac{x^{\alpha}}{2}(e^{i\frac{\alpha\pi}{2}}e_h(ix,y) + e^{-i\frac{\alpha\pi}{2}}e_h(-ix,y)).\end{aligned}$$

Using the fact that $e^{\pm i\frac{\alpha\pi}{2}} = \cos(\alpha\pi/2) \pm i\sin(\alpha\pi/2)$, and transforming the previous term, we get the first equality:

$$\begin{aligned}&\Delta^{\alpha}_{y,h}\cos_h(x,y)\\ &= \frac{x^{\alpha}}{2}\Big(\cos\frac{\alpha\pi}{2}(e_h(ix,y) + e_h(-ix,y)) + i\sin\frac{\alpha\pi}{2}(e_h(ix,y) - e_h(-ix,y))\Big)\\ &= x^{\alpha}\Big(\cos\frac{\alpha\pi}{2}\cos_h(x,y) - \sin\frac{\alpha\pi}{2}\sin_h(x,y)\Big).\end{aligned}$$

The second equality can be proven in the same manner. ∎

Taking $\alpha = 1$ and $\alpha = 2$ in the previous theorem, we get the difference equations likewise the corresponding ones in the classical trigonometry.

Corollary 3.1. *For the functions $y \mapsto \cos_h(x,y)$ and $y \mapsto \sin_h(x,y)$ the following is valid:*

$$\Delta_{y,h}\cos_h(x,y) = -x\sin_h(x,y), \qquad \Delta_{y,h}\sin_h(x,y) = x\cos_h(x,y).$$

Moreover, both functions satisfy the following difference equation:

$$\Delta^2_{y,h}f(y) + x^2 f(y) = 0.$$

Remark 1. When $h \to 0$, previous difference equation becomes

$$\frac{\partial^2}{\partial y^2} f(y) + x^2 f(y) = 0,$$

whose particular solutions are functions $y \mapsto \cos xy$ and $y \mapsto \sin xy$.

Let us consider how the differential operators act on the deformed trigonometric functions.

Lemma 3.1. *For the functions $x \mapsto \cos_h(x,y)$ and $x \mapsto \sin_h(x,y)$ the following equalities hold:*

$$\begin{aligned} \Big((1+h^2x^2)\frac{\partial}{\partial x} - hxy\Big)\cos_h(x,y) &= -y\sin_h(x,y), & (14)\\ \Big((1+h^2x^2)\frac{\partial}{\partial x} - hxy\Big)\sin_h(x,y) &= y\cos_h(x,y). & (15) \end{aligned}$$

P r o o f. Immediately from the definition of the function $e_h(x,y)$, we have

$$\begin{aligned} (1+h^2x^2)\frac{\partial}{\partial x} e_h(ix,y) &= iy(1-ihx)e_h(ix,y),\\ (1+h^2x^2)\frac{\partial}{\partial x} e_h(-ix,y) &= -iy(1+ihx)e_h(-ix,y). \end{aligned}$$

By summing the previous equalities we get

$$\begin{aligned} &(1+h^2x^2)\frac{\partial}{\partial x}\cos_h(x,y) = \frac{iy}{2}((1-ihx)e_h(ix,y) - (1+ihx)e_h(-ix,y))\\ &= \frac{y}{2}\Big(hx(e_h(ix,y)+e_h(-ix,y)) + i(e_h(ix,y) - e_h(-ix,y))\Big)\\ &= y(hx\cos_h(x,y) - \sin_h(x,y)), \end{aligned}$$

wherefrom follows equality (14). Equality (15) can be obtained by subtracting the same equalities. ∎

Theorem 3.2. *The functions $x \mapsto \cos_h(x,y)$ and $x \mapsto \sin_h(x,y)$ satisfy the following differential equation:*

$$\Big((1+h^2x^2)\frac{\partial}{\partial x} - hxy\Big)^2 f(x) + y^2 f(x) = 0. \qquad (16)$$

P r o o f. The statement follows when we apply Lemma 3.1 twice. ∎

Remark 2. Notice that the functions $x \mapsto \cos_h(x,y)$ and $x \mapsto \sin_h(x,y)$ have the same properties with respect to the generalized differential operator $D_h = \Big((1+h^2x^2)\frac{\partial}{\partial x} - hxy\Big)$ as the usual trigonometric functions $x \mapsto \cos xy$ and $x \mapsto \sin xy$ with respect to $\partial/\partial x$. When $h \to 0$, the previous equation becomes

$$\frac{\partial^2}{\partial x^2} f(x) + y^2 f(x) = 0,$$

whose solutions are $x \mapsto \cos xy$ and $x \mapsto \sin xy$.

At last, we give one more differential property of the introduced functions.

Theorem 3.3. *For the functions $(x,y) \mapsto \cos_h(x,y)$ and $(x,y) \mapsto \sin_h(x,y)$ the following equalities hold:*

$$\frac{\partial}{\partial x}\cos_h(x,y) = -y\sin_h(x,y-h), \qquad \frac{\partial}{\partial x}\sin_h(x,y) = y\cos_h(x,y-h).$$

Moreover, both functions satisfy the equation

$$\frac{\partial^2}{\partial x^2} f(x,y) + y^{(2,h)} f(x,y-2h) = 0.$$

Proof. From the definition of $e_h(x,y)$ we have

$$\frac{\partial}{\partial x} e_h(\pm ix, y) = \pm iy e_h(\pm ix, y-h).$$

By addition and subtraction these equalities we get the first partial derivatives of the functions $\cos_h(x,y)$ and $\sin_h(x,y)$. The equation of the second order follows by double applying the operator $\partial/\partial x$ on both functions. ■

Remark 3. When $h \to 0$, the previous equation becomes

$$\frac{\partial^2}{\partial x^2} f(x,y) + y^2 f(x,y) = 0.$$

.

4. Series expansions and graphics interpretations

Having in mind the expansions of the function $e_h(x,y)$ and various deformations of variables, we get several expansions of the deformed trigonometric functions.

Theorem 4.1. *For $x \in \mathbb{C}$ ($|hx| < 1$) and $y \in \mathbb{R}$, the following is valid:*

$$\cos_h(x,y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} y^{(2n,h)}, \quad \sin_h(x,y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} y^{(2n+1,h)}.$$

P r o o f. The expansions follow immediately from the equalities (12) and (13) and the first expansion in (9). ■

In the sequel we deal with real variables, i.e. with $x, y \in \mathbb{R}$. At first, let us propose a new deformation of the variable $x \in \mathbb{R}$ in the following way:

$$x \mapsto \langle x \rangle_h = \frac{1}{h} \arctan hx. \tag{17}$$

Using this deformation, the connection between the deformed and the classical trigonometric functions can be established, as it was done for the exponential functions in (2).

Lemma 4.1. *For $x, y \in \mathbb{R}$ the following holds:*

$$\begin{aligned} \cos_h(x,y) &= e_h(hx^2, y/2) \cos(\langle x \rangle_h y), && (18)\\ \sin_h(x,y) &= e_h(hx^2, y/2) \sin(\langle x \rangle_h y). && (19) \end{aligned}$$

P r o o f. Because of $1 \pm ihx = \sqrt{1+h^2x^2}\, e^{\pm i \arctan hx}$, and (3), we have

$$\begin{aligned} \{\pm ix\}_h &= \frac{1}{h} \ln(1 \pm ihx) = \frac{1}{2h} \ln(1 + h^2x^2) \pm i \frac{1}{h} \arctan hx \\ &= \frac{1}{2} \{hx^2\}_h \pm i \langle x \rangle_h. \end{aligned}$$

Hence, according to (2), we have

$$\begin{aligned} e_h(\pm ix, y) &= e^{\{\pm ix\}_h y} = e^{\frac{y}{2}\{hx^2\}_h \pm i \langle x \rangle_h y} \\ &= e_h(hx^2, y/2)\Big(\cos(\langle x \rangle_h y) \pm i \sin(\langle x \rangle_h y)\Big), \end{aligned}$$

wherefrom we get the required equalities. ■

Theorem 4.2. *For the functions $(x,y) \mapsto \cos_h(x,y)$ and $(x,y) \mapsto \sin_h(x,y)$ the following expansions are valid:*

$$\cos_h(x,y) = \sum_{n=0}^{\infty} C_{n,h}(x)\, y^n, \qquad \sin_h(x,y) = \sum_{n=1}^{\infty} S_{n,h}(x)\, y^n,$$

where

$$
\begin{aligned}
C_{2k,h}(x) &= \frac{(-1)^k}{(2k)!}\sum_{j=0}^{k}\frac{(-1)^j}{2^{2j}}\binom{2k}{2j}\{hx^2\}_h^{2j}\langle x\rangle_h^{2k-2j},\\
C_{2k+1,h}(x) &= \frac{(-1)^k}{(2k+1)!}\sum_{j=0}^{k}\frac{(-1)^j}{2^{2j+1}}\binom{2k+1}{2j+1}\{hx^2\}_h^{2j+1}\langle x\rangle_h^{2k-2j},\\
S_{2k,h}(x) &= \frac{(-1)^k}{(2k)!}\sum_{j=1}^{k}\frac{(-1)^j}{2^{2j-1}}\binom{2k}{2j-1}\{hx^2\}_h^{2j-1}\langle x\rangle_h^{2k-2j+1},\\
S_{2k+1,h}(x) &= \frac{(-1)^k}{(2k+1)!}\sum_{j=0}^{k}\frac{(-1)^j}{2^{2j}}\binom{2k+1}{2j}\{hx^2\}_h^{2j}\langle x\rangle_h^{2k+1-2j}.
\end{aligned}
$$

P r o o f. Using the well–known expansions of the trigonometric functions, equalities (18) and (19) become

$$
\begin{aligned}
\cos_h(x,y) &= e_h(hx^2, y/2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\langle x\rangle_h^{2n}y^{2n},\\
\sin_h(x,y) &= e_h(hx^2, y/2)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\langle x\rangle_h^{2n+1}y^{2n+1}.
\end{aligned}
$$

If we use the second expansion in (9) and arrange the sums, we get the required expressions of the coefficients. ∎

In the previous theorem, the coefficients in the expansions of the deformed trigonometric functions have a convolution form. However, if we take the sums in the closed forms, we can recognize the deformed trigonometric functions in coefficients again.

Theorem 4.3. *The coefficients in the expansions*

$$
\cos_h(x,y) = \sum_{n=0}^{\infty} C_{n,h}(x)\, y^n, \qquad \sin_h(x,y) = \sum_{n=1}^{\infty} S_{n,h}(x)\, y^n,
$$

can be expressed in the following form:

$$
\begin{aligned}
C_{2k,h}(x) &= \frac{(-1)^k}{(2k)!}\langle x\rangle_h^{2k}\cos_h\left(\frac{\{hx^2\}_h}{2h\langle x\rangle_h}, 2kh\right),\\
C_{2k+1,h}(x) &= \frac{(-1)^k}{(2k+1)!}\langle x\rangle_h^{2k+1}\sin_h\left(\frac{\{hx^2\}_h}{2h\langle x\rangle_h}, (2k+1)h\right),
\end{aligned}
$$

$$S_{2k,h}(x) = \frac{(-1)^k}{(2k)!}\langle x\rangle_h^{2k}\sin_h\left(\frac{\{hx^2\}_h}{2h\langle x\rangle_h}, 2kh\right),$$
$$S_{2k+1,h}(x) = \frac{(-1)^k}{(2k+1)!}\langle x\rangle_h^{2k+1}\cos_h\left(\frac{\{hx^2\}_h}{2h\langle x\rangle_h}, (2k+1)h\right).$$

P r o o f. For getting the closed forms of the coefficients we need the well–known equalities [3]

$$\sum_{j=0}^{[n/2]}\binom{n}{2j}t^j = \frac{(1+\sqrt{t})^n + (1-\sqrt{t})^n}{2}, \quad \sum_{j=0}^{[(n-1)/2]}\binom{n}{2j+1}t^j = \frac{(1+\sqrt{t})^n - (1-\sqrt{t})^n}{2\sqrt{t}}.$$

Taking $t = -\frac{\{hx^2\}_h^2}{4h\langle x\rangle_h^2}$ in the first expression in Theorem 4.2, the sum becomes

$$\sum_{j=0}^{2k}\binom{2k}{2j}(-t^2)^j = \frac{1}{2}\Big((1+it)^{2k} + (1-it)^{2k}\Big) = (1+t^2)^k\cos(2k\arctan t)$$

Having in mind the deformation of variable (17), the definition of deformed exponential function (1) and Lemma 4.1, we obtain the coefficient C_{2k}. The rest of coefficients can be obtained in similar way. ■

Finally, we will give a few graphics interpretations of the deformed trigonometric functions. We will consider the function $\cos_h(x,y)$ only, because the behavior of $\sin_h(x,y)$ is similar.

The functions $x \mapsto \cos_h(x,1)$ and $y \mapsto \cos_h(1,y)$ for several positive and negative values of parameter h are shown in Figure 1 and Figure 2, respectively.

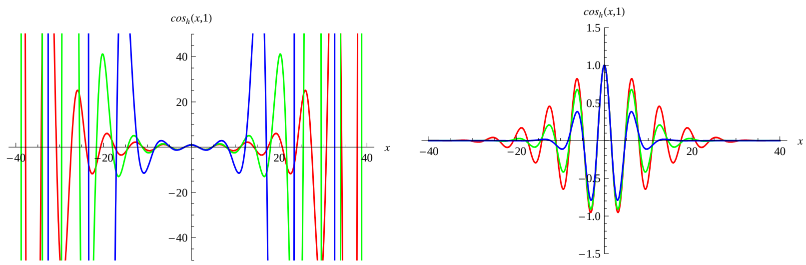

Fig. 1: Deformed trigonometric functions with $y = 1$
for $h = 0.01$, $h = 0.02$, $h = 0.05$ and $h = -0.01$, $h = -0.02$, $h = -0.05$

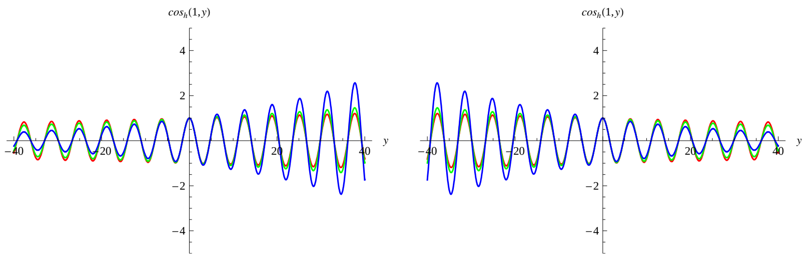

Fig. 2: Deformed trigonometric functions with $x = 1$
for $h = 0.01$, $h = 0.02$, $h = 0.05$ and $h = -0.01$, $h = -0.02$, $h = -0.05$,

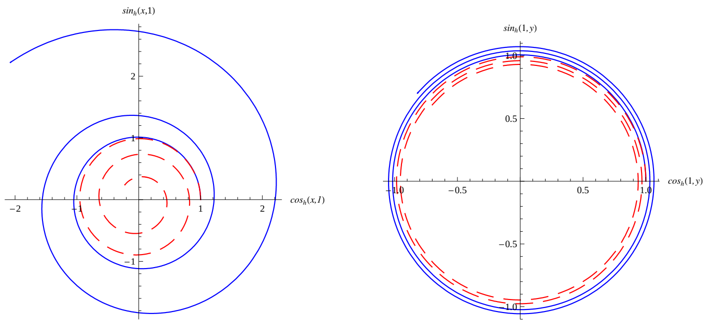

Fig. 3: Parametric representation
$\{\cos_h(x,1), \sin_h(x,1)\}$ and $\{\cos_h(1,y), \sin_h(1,y)\}$
for $h = 0.01$ (continuous curve) and $h = -0.01$ (dashed curve)

At last, the function $(x, y) \mapsto \cos_h(x, y)$ for positive and negative h is shown in Figure 4.

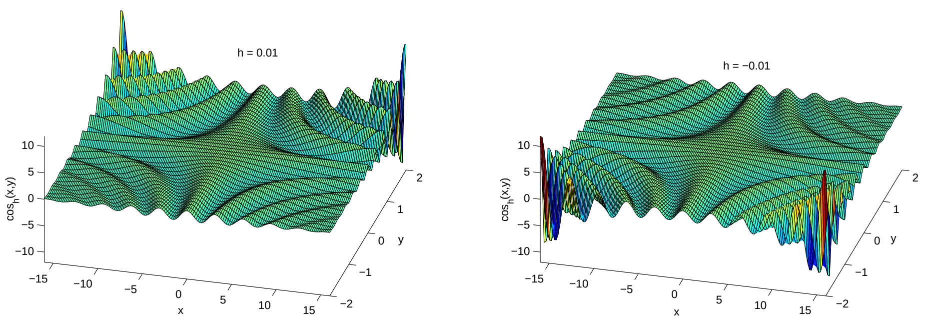

Fig. 4: Deformed trigonometric functions for $h = 0.01$ and $h = -0.01$

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[1,3] *University of Niš, Faculty of Electronic Engineering*
Aleksandra Medvedeva 14
18000 Niš, SERBIA

e-maila: [1] *sladjana.marinkovic@elfak.ni.ac.rs* , [3] *edinmulalic@yahoo.com*
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[2] *University of Niš, Faculty of Occupational Safety*
Čarnojevića 10A
18000 Niš, SERBIA

e-mail: miomir.stankovic@gmail.com